

# ALGEBRAIC ASPECTS OF THE $L_2$ ANALYTIC GAUSSIAN–FOURIER–FEYNMAN TRANSFORM VIA GAUSSIAN PROCESSES ON WIENER SPACE

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**ABSTRACT.** In this research, we investigate several rotation properties of the generalized Wiener integral with respect to Gaussian processes, which are then used to analyze an  $L_2$  analytic Gaussian–Fourier–Feynman transform. Our results indicate that the  $L_2$  analytic Gaussian–Fourier–Feynman transforms are linear operator isomorphisms from a Hilbert space into itself. We then proceed to investigate the algebraic structure of these generalized transforms and establish that two classes of the generalized transforms on Wiener space are group isomorphic.

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## 1. INTRODUCTION

Let  $C_0[0, T]$  be the one parameter Wiener space. Bearman’s rotation theorem [1] for Wiener measure has played an important role in various research areas in mathematics and physics involving Wiener integration theory. Bearman’s theorem was further developed by Cameron and Storvick [3] and by Johnson and Skoug [9] in their studies of Wiener integral equations.

The concept of the generalized Wiener integral with respect to Gaussian paths  $\mathcal{Z}_h(x, \cdot)$  (abbr.  $\mathcal{Z}_h$ -Wiener integral) and the generalized analytic Feynman integral with respect to Gaussian paths  $\mathcal{Z}_h(x, \cdot)$  (abbr.  $\mathcal{Z}_h$ -Feynman integral) on  $C_0[0, T]$  were introduced by Chung, Park and Skoug [4, 13], and further developed in [6, 14].

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In [4, 6, 13, 14], the  $\mathcal{Z}_h$ -Wiener integral is defined by the Wiener integral

$$(1.1) \quad \int_{C_0[0,T]} F(\mathcal{Z}_h(x, \cdot)) dm(x),$$

where  $\mathcal{Z}_h(x, \cdot)$  is the Gaussian path given by the stochastic integral  $\mathcal{Z}_h(x, t) = \int_0^t h(s) dx(s)$  with  $h \in L^2[0, T]$ .

In this paper, we first establish the Bearman-type theorems for the generalized Wiener integral given by (1.1). Using these results, we then take a closer look at the  $L_2$  analytic generalized Fourier–Feynman transform introduced by Huffman, Park, and Skoug in [6]. Our results indicate that the  $L_2$  analytic generalized Fourier–Feynman transforms are linear operator isomorphisms from a Hilbert space of cylinder functionals on Wiener space into itself. The algebraic structures of these generalized transforms are examined. Based on this examination, we show that the two classes of generalized transforms are group isomorphic.

## 2. PRELIMINARIES

Let  $C_0[0, T]$  denote one-parameter Wiener space, that is the space of all real-valued continuous functions  $x$  on  $[0, T]$  with  $x(0) = 0$ . Let  $\mathcal{M}$  denote the class of all Wiener measurable subsets of  $C_0[0, T]$  and let  $m$  denote Wiener measure. Then  $(C_0[0, T], \mathcal{M}, m)$  is a complete measure space.

A subset  $B$  of  $C_0[0, T]$  is said to be scale-invariant measurable [9] provided  $\rho B \in \mathcal{M}$  for all  $\rho > 0$ , and a scale-invariant measurable set  $N$  is said to be scale-invariant null provided  $m(\rho N) = 0$  for all  $\rho > 0$ . A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (abbr. s-a.e.). A functional  $F$  is said to be scale-invariant measurable provided  $F$  is defined on a scale-invariant measurable set and  $F(\rho \cdot)$  is Wiener measurable for every  $\rho > 0$ . If two functionals  $F$  and  $G$  are equal s-a.e., we write  $F \approx G$ .

The Paley-Wiener-Zygmund (abbr. PWZ) stochastic integral [11] plays a key role throughout this paper. For  $v$  in  $L_2[0, T]$ , the PWZ stochastic integral  $\langle v, x \rangle$  is given by the formula

$$\langle v, x \rangle := \lim_{n \rightarrow \infty} \int_0^T \sum_{j=1}^n (v, \phi_j)_2 \phi_j(t) dx(t)$$

for  $m$ -a.e.  $x \in C_0[0, T]$ , where  $\{\phi_n\}$  is a complete orthonormal set of functions of bounded variation on  $[0, T]$  and  $(\cdot, \cdot)_2$  denotes the  $L_2$ -inner product.

It is known that for each  $v \in L_2[0, T]$ , the PWZ integral  $\langle v, x \rangle$  is essentially independent of the choice of the complete orthonormal set  $\{\phi_n\}$ . If  $v$  is of bounded variation on  $[0, T]$  then  $\langle v, x \rangle$  equals the Riemann-Stieltjes integral  $\int_0^T v(t) dx(t)$  for s-a.e.  $x \in C_0[0, T]$ , and for all  $v \in L_2[0, T]$ ,  $\langle v, x \rangle$  is a Gaussian random variable on  $C_0[0, T]$  with mean zero and variance  $\|v\|_2^2$ . For a more detailed study of the PWZ stochastic integral, see [10, 12].

For any  $h \in L_2[0, T]$  with  $\|h\|_2 > 0$ , let  $\mathcal{Z}_h$  be the PWZ stochastic integral

$$(2.1) \quad \mathcal{Z}_h(x, t) := \langle v \chi_{[0,t]}, x \rangle \equiv \int_0^t h(s) dx(s),$$

and let

$$(2.2) \quad \beta_h(t) := \int_0^t h^2(u) du.$$

The stochastic process  $\mathcal{Z}_h$  on  $C_0[0, T] \times [0, T]$  is a Gaussian process with mean zero and covariance function

$$\int_{C_0[0, T]} \mathcal{Z}_h(x, s) \mathcal{Z}_h(x, t) dm(x) = \beta_h(\min\{s, t\}).$$

In addition, by [17, Theorem 21.1],  $\mathcal{Z}_h(\cdot, t)$  is stochastically continuous in  $t$  on  $[0, T]$ , and for any  $h_1, h_2 \in L_2[0, T]$ ,

$$\int_{C_0[0, T]} \mathcal{Z}_{h_1}(x, s) \mathcal{Z}_{h_2}(x, t) dm(x) = \int_0^{\min\{s, t\}} h_1(u) h_2(u) du.$$

From [4, Lemma 1], it follows that for each  $v \in L_2[0, T]$  and  $h \in L_\infty[0, T]$ ,

$$(2.3) \quad \langle v, \mathcal{Z}_h(x, \cdot) \rangle = \langle vh, x \rangle$$

for s-a.e.  $x \in C_0[0, T]$ .

Throughout the rest of this paper let  $\mathbb{C}_+$  and  $\tilde{\mathbb{C}}_+$  denote the set of complex numbers with positive real part and nonzero complex numbers with nonnegative real part, respectively.

Let  $F$  be a complex-valued scale-invariant measurable functional on  $C_0[0, T]$  such that  $J(h; \lambda) := \int_{C_0[0, T]} F(\lambda^{-1/2} \mathcal{Z}_h(x, \cdot)) dm(x)$  exists as a finite number for all  $\lambda > 0$ . If there exists a function  $J^*(h; \lambda)$  analytic on  $\mathbb{C}_+$  such that  $J^*(h; \lambda) = J(h; \lambda)$  for all  $\lambda > 0$ , then  $J^*(h; \lambda)$  is defined to be the analytic  $\mathcal{Z}_h$ -Wiener integral of  $F$  over  $C_0[0, T]$  with parameter  $\lambda$ , and for  $\lambda \in \mathbb{C}_+$  we write

$$\int_{C_0[0, T]}^{\text{anw}_\lambda} F(\mathcal{Z}_h(x, \cdot)) dm(x) := J^*(h; \lambda).$$

Let  $q$  be a nonzero real number and let  $F$  be a functional such that  $I_h^{\text{anw}_\lambda}[F]$  exists for all  $\lambda \in \mathbb{C}_+$ . If the following limit exists, we call it the analytic  $\mathcal{Z}_h$ -Feynman integral of  $F$  with parameter  $q$ , and we write

$$I_h^{\text{anf}_q}[F] \equiv I_{h, x}^{\text{anf}_q}[F(\mathcal{Z}_h(x, \cdot))] := \lim_{\lambda \rightarrow -iq} \int_{C_0[0, T]}^{\text{anw}_\lambda} F(\mathcal{Z}_h(x, \cdot)) dm(x)$$

where  $\lambda$  approaches  $-iq$  through values in  $\mathbb{C}_+$ .

We are now ready to state the definition of the analytic Gaussian-Fourier-Feynman transform (abbr. Gaussian-FFT) on Wiener space.

**Definition 2.1.** Let  $F$  be a scale-invariant measurable functional on  $C_0[0, T]$ , and let  $h \in L_2[0, T]$  with  $\|h\|_2 > 0$  be given. For  $\lambda \in \mathbb{C}_+$  and  $y \in C_0[0, T]$ , let

$$T_{\lambda, h}(F)(y) := \int_{C_0[0, T]}^{\text{anw}_\lambda} F(y + \mathcal{Z}_h(x, \cdot)) dm(x).$$

Let  $q$  be a nonzero real number. We define the  $L_p$  analytic  $\mathcal{Z}_h$ -FFT (Gaussian-FFT with respect to the process  $\mathcal{Z}_h$ ),  $T_{q, h}^{(2)}(F)$  of  $F$ , by the formula ( $\lambda \in \mathbb{C}_+$ ),

$$T_{q, h}^{(2)}(F)(y) := \text{l. i. m.}_{\lambda \rightarrow -iq} T_{\lambda, h}(F)(y)$$

if it exists; i.e., for each  $\rho > 0$ ,

$$\lim_{\lambda \rightarrow -iq} \int_{C_0[0, T]} |T_{\lambda, h}(F)(\rho y) - T_{q, h}^{(2)}(F)(\rho y)|^2 dm(y) = 0.$$

We note that  $T_{q,h}^{(2)}(F)$  is defined only s-a.e.. We also note that if  $T_{q,h}^{(2)}(F)$  exists and if  $F \approx G$ , then  $T_{q,h}^{(2)}(G)$  exists and  $T_{q,h}^{(2)}(G) \approx T_{q,h}^{(2)}(F)$ . One can see that for each  $h \in L_2[0, T]$ ,  $T_{q,h}^{(2)}(F) \approx T_{q,-h}^{(2)}(F)$  since

$$(2.4) \quad \int_{C_0[0,T]} F(x) dm(x) = \int_{C_0[0,T]} F(-x) dm(x).$$

*Remark 2.2.* Note that if  $h \equiv 1$  on  $[0, T]$ , then the definition of the  $L_2$  analytic  $\mathcal{Z}_1$ -FFT agrees with the previous definition of the analytic Fourier–Feynman transform [2, 5, 8].

### 3. ROTATION OF WIENER MEASURES WITH RESPECT TO GAUSSIAN PATHS

In this section we establish rotation theorems for generalized Wiener integral with respect to Gaussian paths. Throughout this section, we will assume that each functional  $F : C_0[0, T] \rightarrow \mathbb{R}$  we consider is scale-invariant measurable and that

$$\int_{C_0[0,T]} |F(\rho \mathcal{Z}_h(x, \cdot))| dm(x) < +\infty$$

for each  $\rho > 0$  and all  $h \in L_2[0, T]$ .

Let  $m_L$  denote the Lebesgue measure on  $[0, T]$ . Let  $v_1$  and  $v_2$  be functions in  $L_2[0, T]$  with  $\|v_1\|_2 = \|v_2\|_2 \equiv \sigma^2 > 0$ . The random variables  $X_1(x) = \langle v_1, x \rangle$  and  $X_2(x) = \langle v_2, x \rangle$  will then have the same distribution,  $N(0, \sigma^2)$ .

Next let  $h_1$  and  $h_2$  be functions in  $L_2[0, T]$  with  $\|h_j\|_2 > 0$ ,  $j \in \{1, 2\}$ . Then there exists a function  $\mathbf{s} \in L_2[0, T]$  such that

$$(3.1) \quad \mathbf{s}^2(t) = h_1^2(t) + h_2^2(t)$$

for  $m_L$ -a.e.  $t \in [0, T]$ . We note that for any  $h_1 \in L_2[0, T]$ , there exists  $h_2 \in L_2[0, T]$  such that  $\|h_1\|_2 = \|h_2\|_2$ , but,  $h_1 \neq h_2$  for  $m_L$ -a.e. on  $[0, T]$ . Thus one can see that the function ‘ $\mathbf{s}$ ’ satisfying (3.1) is not unique. Let  $\mathbf{s}_1$  and  $\mathbf{s}_2$  be functions in  $L_2[0, T]$  that satisfy equation (3.1). Then, in view of the observation above,  $\langle \mathbf{s}_1, x \rangle$  and  $\langle \mathbf{s}_2, x \rangle$  have the same distribution  $N(0, \|h_1\|_2^2 + \|h_2\|_2^2)$ . We will use the symbol  $\mathbf{s}(h_1, h_2)$  for the functions ‘ $\mathbf{s}$ ’ that satisfy (3.1) above. If  $h_1$  and  $h_2$  are functions of  $L_2[0, T]$  (resp.  $L_\infty[0, T]$ ), then infinitely many functions,  $\mathbf{s}(h_1, h_2)$ , exists in  $L_2[0, T]$  (resp.  $L_\infty[0, T]$ ).

Let  $\mathcal{H} = (h_1, \dots, h_n)$  be a finite sequence of functions in  $L_2[0, T]$ . For each  $k \in \{1, \dots, n\}$ , let  $\mathbf{s}(h_1, \dots, h_k)$  be an element of  $L_2[0, T]$  such that

$$(3.2) \quad \mathbf{s}^2(h_1, \dots, h_k)(t) = \sum_{j=1}^k h_j^2(t)$$

for  $m_L$ -a.e.  $t \in [0, T]$ . Then for  $k \in \{1, \dots, n-1\}$ ,

$$\begin{aligned} \mathbf{s}^2(\mathbf{s}(h_1, \dots, h_k), h_{k+1})(t) &= \mathbf{s}^2(h_1, \dots, h_k)(t) + h_{k+1}^2(t) = \sum_{j=1}^{k+1} h_j^2(t) \\ &= \mathbf{s}^2(h_1, \dots, h_k, h_{k+1})(t) \end{aligned}$$

for  $m_L$ -a.e.  $t \in [0, T]$ . From these we have the following properties:

(i) For any permutation  $\pi$  of  $I_n = \{1, 2, \dots, n\}$ ,

$$(3.3) \quad \mathbf{s}(\mathcal{H}) \equiv \mathbf{s}(h_1, h_2, \dots, h_n) = \mathbf{s}(h_{\pi(1)}, h_{\pi(2)} \dots, h_{\pi(n)})$$

for  $m_L$ -a.e.  $t \in [0, T]$ .

(ii) Given two sequences  $\mathcal{H}_1 = (h_{11}, h_{12}, \dots, h_{1n_1})$  and  $\mathcal{H}_2 = (h_{21}, h_{22}, \dots, h_{2n_2})$  of functions in  $L_2[0, T]$ , let

$$(3.4) \quad \mathcal{H}_1 \wedge \mathcal{H}_2 := (h_{11}, h_{12}, \dots, h_{1n_1}, h_{21}, h_{22}, \dots, h_{2n_2}).$$

Then

$$(3.5) \quad \mathbf{s}(\mathcal{H}_1 \wedge \mathcal{H}_2) \equiv \mathbf{s}(h_{11}, h_{12}, \dots, h_{1n_1}, h_{21}, h_{22}, \dots, h_{2n_2}) = \mathbf{s}(\mathbf{s}(\mathcal{H}_1), \mathbf{s}(\mathcal{H}_2)).$$

For  $h_1, h_2 \in L_2[0, T]$  with  $\|h_j\|_2 > 0$ ,  $j \in \{1, 2\}$ , let  $\mathcal{Z}_{h_1}$  and  $\mathcal{Z}_{h_2}$  be the Gaussian processes given by (2.1) with  $h$  replaced with  $h_1$  and  $h_2$  respectively. Then the process

$$\mathfrak{Z}_{h_1, h_2} : C_0[0, T] \times C_0[0, T] \times [0, T] \rightarrow \mathbb{R}$$

given by

$$\mathfrak{Z}_{h_1, h_2}(x_1, x_2, t) := \mathcal{Z}_{h_1}(x_1, t) + \mathcal{Z}_{h_2}(x_2, t)$$

is also a Gaussian process with mean zero and covariance

$$\begin{aligned} & \int_{C_0^2[0, T]} \mathfrak{Z}_{h_1, h_2}(x_1, x_2, s) \mathfrak{Z}_{h_1, h_2}(x_1, x_2, t) dm^2(x_1, x_2) \\ &= \beta_{h_1}(\min\{s, t\}) + \beta_{h_2}(\min\{s, t\}) \end{aligned}$$

where  $\beta_h$  is given by (2.2) above.

Next we consider the stochastic process  $\mathcal{Z}_{\mathbf{s}(h_1, h_2)} : C_0[0, T] \times [0, T] \rightarrow \mathbb{R}$ . As stated in Section 2 above, the process  $\mathcal{Z}_{\mathbf{s}(h_1, h_2)}$  is Gaussian with mean zero and covariance

$$\begin{aligned} \beta_{\mathbf{s}(h_1, h_2)}(\min\{s, t\}) &= \int_0^{\min\{s, t\}} [\mathbf{s}(h_1, h_2)(u)]^2 du \\ &= \int_0^{\min\{s, t\}} (h_1^2(u) + h_2^2(u)) du \\ &= \beta_{h_1}(\min\{s, t\}) + \beta_{h_2}(\min\{s, t\}). \end{aligned}$$

From these facts, one can see that  $\mathfrak{Z}_{h_1, h_2}$  and  $\mathcal{Z}_{\mathbf{s}(h_1, h_2)}$  have the same distribution. Thus we obtain the following theorems and corollaries.

**Theorem 3.1.** *Let  $F$  be a functional on  $C_0[0, T]$ , and let  $h_1$  and  $h_2$  be elements of  $L_2[0, T]$ . Then*

$$(3.6) \quad \begin{aligned} & \int_{C_0^2[0, T]} F(\mathcal{Z}_{h_1}(x_1, \cdot) + \mathcal{Z}_{h_2}(x_2, \cdot)) dm^2(x_1, x_2) \\ &= \int_{C_0[0, T]} F(\mathcal{Z}_{\mathbf{s}(h_1, h_2)}(x, \cdot)) dm(x). \end{aligned}$$

**Corollary 3.2.** *Let  $F$  be a functional on  $C_0[0, T]$ , and let  $\mathcal{H} = (h_1, h_2, \dots, h_n)$  be a sequence of functions in  $L_2[0, T]$ . Then*

$$\int_{C_0^n[0, T]} F\left(\sum_{j=1}^n \mathcal{Z}_{h_j}(x_j, \cdot)\right) dm^n(\vec{x}) = \int_{C_0[0, T]} F(\mathcal{Z}_{\mathbf{s}(h_1, \dots, h_n)}(x, \cdot)) dm(x).$$

**Theorem 3.3.** *Let  $F$  be a functional on  $C_0[0, T]$ , and let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be finite sequences of functions in  $L_2[0, T]$ . Then*

$$\begin{aligned}
 (3.7) \quad & \int_{C_0^2[0, T]} F(\mathcal{Z}_{\mathbf{s}(\mathcal{H}_1)}(x_1, \cdot) + \mathcal{Z}_{\mathbf{s}(\mathcal{H}_2)}(x_2, \cdot)) dm^2(x_1, x_2) \\
 &= \int_{C_0[0, T]} F(\mathcal{Z}_{\mathbf{s}(\mathcal{H}_1, \mathcal{H}_2)}(x, \cdot)) dm(x) \\
 &= \int_{C_0[0, T]} F(\mathcal{Z}_{\mathbf{s}(\mathcal{H}_1 \wedge \mathcal{H}_2)}(x, \cdot)) dm(x).
 \end{aligned}$$

**Corollary 3.4.** *Let  $F$  be a functional on  $C_0[0, T]$ , and let  $\mathcal{H} = (h_1, h_2, \dots, h_n)$  be a sequence of functions in  $L_2[0, T]$  and let  $h_{n+1}$  be an element of  $L_2[0, T]$ . Then*

$$\begin{aligned}
 (3.8) \quad & \int_{C_0^2[0, T]} F(\mathcal{Z}_{\mathbf{s}(\mathcal{H})}(x_1, \cdot) + \mathcal{Z}_{h_{n+1}}(x_2, \cdot)) dm^2(x_1, x_2) \\
 &= \int_{C_0[0, T]} F(\mathcal{Z}_{\mathbf{s}(\mathcal{H}, h_{n+1})}(x, \cdot)) dm(x) \\
 &= \int_{C_0[0, T]} F(\mathcal{Z}_{\mathbf{s}(\mathcal{H} \wedge (h_{n+1}))}(x, \cdot)) dm(x).
 \end{aligned}$$

*Remark 3.5.* Using seminal results by Bearman [1], Cameron and Storvick [3] established a rotation property of the Wiener measure  $m$ . The result is summarized as follows: for a Wiener-integrable functional  $F$  and every nonzero real  $a$  and  $b$ ,

$$(3.9) \quad \int_{C_0^2[0, T]} F(ax + by) d(m \times m)(x, y) \stackrel{*}{=} \int_{C_0[0, T]} F(\sqrt{a^2 + b^2} z) dm(z),$$

where by  $\stackrel{*}{=}$  we mean that if either side exists, both sides exist and equality holds.

Let  $h_1 \equiv a$  and  $h_2 \equiv b$ , as constant functions on  $[0, T]$  in (3.6) above. Then in view of equation (3.1), we see that either  $\mathbf{s}(a, b) = \sqrt{a^2 + b^2}$  or  $\mathbf{s}(a, b) = -\sqrt{a^2 + b^2}$ . Thus, in view of equation (2.4), we obtain equation (3.9) as a special case of (3.6).

#### 4. $L_2$ ANALYTIC GAUSSIAN-FOURIER-FEYNMAN TRANSFORM

In this section, we analyze the analytic Gaussian-FFT of cylinder functionals. Functionals that involve PWZ stochastic integrals are quite common. A functional  $F$  on  $C_0[0, T]$  is called a cylinder functional if there exists a linearly independent subset  $\{v_1, \dots, v_m\}$  of functions in  $L_2[0, T]$  such that

$$(4.1) \quad F(x) = \psi(\langle v_1, x \rangle, \dots, \langle v_m, x \rangle), \quad x \in C_0[0, T],$$

where  $\psi$  is a complex-valued Lebesgue measurable function on  $\mathbb{R}^m$ .

It is easy to show that for the given cylinder functional  $F$  of the form (4.1), there exists an orthogonal subset  $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$  of  $L_2[0, T]$  such that  $F$  can be expressed as

$$(4.2) \quad F(x) = f(\langle \alpha_1, x \rangle, \dots, \langle \alpha_n, x \rangle), \quad x \in C_0[0, T],$$

where  $f$  is a complex-valued Lebesgue measurable function on  $\mathbb{R}^n$ . Thus, we lose no generality in assuming that every cylinder functional on  $C_0[0, T]$  is of the form (4.2).

For  $h \in L_\infty[0, T]$  with  $\|h\|_2 > 0$ , let  $\mathcal{Z}_h$  be the Gaussian process given by (2.1) above and let  $F$  be given by equation (4.2). Then by equation (2.3),

$$\begin{aligned} F(\mathcal{Z}_h(x, \cdot)) &= f(\langle \alpha_1, \mathcal{Z}_h(x, \cdot) \rangle, \dots, \langle \alpha_n, \mathcal{Z}_h(x, \cdot) \rangle) \\ &= f(\langle \alpha_1 h, x \rangle, \dots, \langle \alpha_n h, x \rangle). \end{aligned}$$

Even though the set  $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$  of functions in  $L_2[0, T]$  is orthogonal, the subset  $\mathcal{A}h \equiv \{\alpha_1 h, \dots, \alpha_n h\}$  of  $L_2[0, T]$  need not be orthogonal.

Let  $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$  be an orthogonal subset of  $L_2[0, T]$  which has no zero, and let  $\mathcal{O}_\infty(\mathcal{A})$  denote the class of all nonzero elements  $h \in L_\infty[0, T]$  such that  $\mathcal{A}h$  is orthogonal in  $L_2[0, T]$ . Since  $\dim L_2[0, T] = \infty$ , infinitely many functions,  $h$ , exist in  $\mathcal{O}_\infty(\mathcal{A})$ .

**Example 4.1.** For any  $\rho \in \mathbb{R} \setminus \{0\}$ ,  $\rho \in \mathcal{O}_\infty(\mathcal{A})$ , as a constant function on  $[0, T]$ .

**Example 4.2.** For any orthogonal set  $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$  of functions in  $L_2[0, T]$ , each of whose element is of bounded variation on  $[0, T]$ , let  $L(S)$  be the subspace of  $L_2[0, T]$  which is spanned by  $S = \{\alpha_i \alpha_j : 1 \leq i < j \leq n\}$ , and let  $L(S)^\perp$  be the orthogonal complement of  $L(S)$ . Let

$$\mathcal{P}_\infty(\mathcal{A}) := \{h \in L_\infty[0, T] : h^2 \in L(S)^\perp \text{ and } \|h\|_2 > 0\}.$$

Since  $\dim L(S)$  is finite, and  $L_\infty[0, T]$  is dense in  $L_2[0, T]$ ,  $\dim(L(S)^\perp \cap L_\infty[0, T]) = \infty$  and so  $\mathcal{P}_\infty(\mathcal{A})$  has infinitely many elements.

Let  $h$  be an element of  $\mathcal{P}_\infty(\mathcal{A})$ . It is easy to show that  $\|\alpha_j h\|_2 > 0$  for all  $j \in \{1, \dots, n\}$ . From the definition of the  $\mathcal{P}_\infty(\mathcal{A})$ , we see that for  $i, j \in \{1, \dots, n\}$  with  $i \neq j$ ,

$$(\alpha_i h, \alpha_j h)_2 = \int_0^T \alpha_i(t) \alpha_j(t) h^2(t) dt = 0.$$

From these, we see that  $\mathcal{A}h$  is an orthogonal set in  $L_2[0, T]$  for any  $h$  in  $\mathcal{P}_\infty(\mathcal{A})$ , i.e.,  $\mathcal{P}_\infty(\mathcal{A}) \subset \mathcal{O}_\infty(\mathcal{A})$ .

**Example 4.3.** For each  $j \in \{1, 2, \dots\}$ , let  $e_j(t) = \cos\left(\frac{(j-1/2)\pi}{T}t\right)$  on  $[0, T]$ . Then,  $\Gamma \equiv \{e_j\}_{j=1}^\infty$  is an orthogonal sequence of functions in  $L_2[0, T]$ . In fact,  $e_j$  is of bounded variation on  $[0, T]$  for every  $j \in \{1, 2, \dots\}$ . Let  $\mathcal{A}$  be any finite subset  $\{e_{j_1}, \dots, e_{j_n}\}$  of  $\Gamma$  with  $j_1 < \dots < j_n$ . Then, we see that for all  $k > j_n$  and all  $\rho \in \mathbb{R} \setminus \{0\}$ ,  $\rho e_k$  is an element of  $\mathcal{O}_\infty(\mathcal{A})$ . Moreover, for all  $k > j_n$ ,  $\mathcal{A}\rho e_k$  with  $\rho = T/2$  is an orthonormal set of functions in  $L_2[0, T]$ . It is well known that  $\Gamma' \equiv \{\sqrt{2/T}e_j\}_{j=1}^\infty$  is a complete orthonormal set of functions in  $L_2[0, T]$ .

Given  $h_1$  and  $h_2$  in  $\mathcal{O}_\infty(\mathcal{A})$ , let  $\mathbf{s}(h_1, h_2)$  be an element of  $L_\infty[0, T]$  which satisfies equation (3.1) above. Then we observe that for all  $j, l \in \{1, \dots, n\}$  with  $j \neq l$ ,

$$\begin{aligned} (\alpha_j \mathbf{s}(h_1, h_2), \alpha_l \mathbf{s}(h_1, h_2))_2 &= \int_0^T \alpha_j(t) \alpha_l(t) [\mathbf{s}(h_1, h_2)(t)]^2 dt \\ &= \int_0^T \alpha_j(t) \alpha_l(t) (h_1^2(t) + h_2^2(t)) dt \\ &= \int_0^T \alpha_j(t) \alpha_l(t) h_1^2(t) dt + \int_0^T \alpha_j(t) \alpha_l(t) h_2^2(t) dt \\ &= (\alpha_j h_1, \alpha_l h_1)_2 + (\alpha_j h_2, \alpha_l h_2)_2 = 0 \end{aligned} \tag{4.3}$$

and that for each  $j \in \{1, \dots, n\}$ ,

$$(4.4) \quad \begin{aligned} \|\alpha_j \mathbf{s}(h_1, h_2)\|_2^2 &= \int_0^T \alpha_j^2 h_1^2(t) dt + \int_0^T \alpha_j^2 h_2^2(t) dt \\ &= \|\alpha_j h_1\|_2^2 + \|\alpha_j h_2\|_2^2. \end{aligned}$$

Hence, from (4.3) and (4.4), we see that  $\mathcal{A}\mathbf{s}(h_1, h_2) = \{\alpha_1 \mathbf{s}(h_1, h_2), \dots, \alpha_n \mathbf{s}(h_1, h_2)\}$  is an orthogonal set of functions in  $L_2[0, T]$  and that the PWZ stochastic integrals

$$\langle \alpha_j, \mathcal{Z}_{\mathbf{s}(h_1, h_2)}(x, \cdot) \rangle = \langle \alpha_j \mathbf{s}(h_1, h_2), x \rangle, \quad j \in \{1, \dots, n\}$$

form a set of independent Gaussian random variables on  $C_0[0, T]$ .

Given an orthogonal set  $\mathcal{A}$  in  $L_2[0, T]$ , let  $\mathfrak{A}^{(2)}$  be the space of all functionals  $F : C_0[0, T] \rightarrow \mathbb{C}$  of the form (4.2) for s-a.e.  $x \in C_0[0, T]$ , where  $f : \mathbb{R}^n \rightarrow \mathbb{C}$  is in  $L_2(\mathbb{R}^n)$ . Note that  $F \in \mathfrak{A}^{(2)}$  implies that  $F$  is scale-invariant measurable.

Throughout this and next sections, for convenience, we use the following notation: for a finite subset  $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$  of functions in  $L_2[0, T]$ ,  $f \in L_2(\mathbb{R}^n)$ ,  $q \in \mathbb{R} \setminus \{0\}$  and  $h \in L_\infty[0, T]$ , let

$$(4.5) \quad \begin{aligned} \psi_{f, \mathcal{A}h}^q(\vec{r}) &\equiv \psi_{f, \mathcal{A}h}^q(\mathcal{A}h; r_1, \dots, r_n) \\ &:= \left( \prod_{j=1}^n \frac{-iq}{2\pi \|\alpha_j h\|_2^2} \right)^{1/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp \left\{ \frac{iq}{2} \sum_{j=1}^n \frac{(u_j - r_j)^2}{\|\alpha_j h\|_2^2} \right\} d\vec{u}. \end{aligned}$$

Note that  $\mathfrak{A}^{(2)}$  is a rich class of functionals, because it contains many unbounded functionals. Also, it contains the Schwartz class  $\mathcal{S}(\mathbb{R}^n)$ . The functionals of class  $\mathcal{S}(\mathbb{R}^n)$  are of interest in Feynman integration theory and quantum mechanics.

For  $F_1$  and  $F_2$  in  $\mathfrak{A}_n^{(2)}$ , let

$$\langle\langle F_1, F_2 \rangle\rangle_{\mathfrak{A}^{(2)}} := \int_{\mathbb{R}^n} f_1(\vec{u}) \overline{f_2(\vec{u})} d\vec{u}$$

denote the inner product on  $\mathfrak{A}^{(2)}$ , where  $f_j$ ,  $j \in \{1, 2\}$ , is the corresponding function of  $F_j$  by equation (4.2), and let  $\|F\|_{\mathfrak{A}^{(2)}} := \langle\langle F, F \rangle\rangle_{\mathfrak{A}^{(2)}}^{1/2}$ . Then one can see that  $(\mathfrak{A}_n^{(2)}, \|\cdot\|_{\mathfrak{A}_n^{(2)}})$  is a complex normed linear space. Also, from the fact that  $L_2(\mathbb{R}^n)$  is complete, it can be easily shown that the space  $(\mathfrak{A}_n^{(2)}, \|\cdot\|_{\mathfrak{A}_n^{(2)}})$  is a Hilbert space.

In [5], Huffman, Park, and Skoug established the existence of the  $L_2$  analytic  $\mathcal{Z}_1$ -FFT for cylinder functionals having the form (4.2). The following theorem is a development of Theorem 2.2 in [5].

**Theorem 4.4.** *Let  $F \in \mathfrak{A}^{(2)}$  be given by equation (4.2) and let  $h$  be an element of  $\mathcal{O}_\infty(\mathcal{A})$ . Then*

- (i) *for all nonzero real  $q \in \mathbb{R} \setminus \{0\}$ , the  $L_2$  analytic  $\mathcal{Z}_h$ -FFT of  $F$ ,  $T_{q,h}^{(2)}(F)$ , exists, belongs to  $\mathfrak{A}^{(2)}$  and is given by the formula*

$$T_{q,h}^{(2)}(F)(y) = \psi_{f, \mathcal{A}h}^q(\langle \alpha_1 h, x \rangle, \dots, \langle \alpha_n h, x \rangle)$$

*for s-a.e.  $y \in C_0[0, T]$ , where  $\psi_{f, \mathcal{A}h}^q$  is given by (4.5); and*

- (ii) *for all nonzero real  $q \in \mathbb{R} \setminus \{0\}$ ,*

$$T_{-q,h}^{(2)}(T_{q,h}^{(2)}(F)) \approx F.$$



In other words, the  $L_2$  analytic  $\mathcal{Z}_h$ -FFT,  $T_{q,h}^{(2)}$  has the inverse transform  $\{T_{q,h}^{(2)}\}^{-1} = T_{-q,h}^{(2)}$ .

**Theorem 4.5.** Let  $F \in \mathfrak{A}^{(2)}$  be given by equation (4.2), and let  $h_1$  and  $h_2$  be elements of  $\mathcal{O}_\infty(\mathcal{A})$ . Then for all nonzero real  $q \in \mathbb{R} \setminus \{0\}$ ,

$$(4.6) \quad T_{q,h_2}^{(2)}(T_{q,h_1}^{(2)}(F)) = T_{q,s(h_1,h_2)}^{(2)}(F)(y)$$

for s-a.e.  $y \in C_0[0, T]$ .

*Proof.* In view of Theorem 4.4, the two analytic Gaussian-FFTs in equation (4.6) exist. Thus equality is what needs to be shown. Hence, to establish equation (4.6), it will suffice to show that for each  $\lambda > 0$ ,

$$T_{\lambda,h_2}(T_{\lambda,h_1}(F))(y) = T_{\lambda,s(h_1,h_2)}(F)(y)$$

for s-a.e.  $y \in C_0[0, T]$ . But, using equation (3.6), for each  $\lambda > 0$  and s-a.e.  $y \in C_0[0, T]$ , we obtain

$$\begin{aligned} & T_{\lambda,h_2}(T_{\lambda,h_1}(F))(y) \\ &= \int_{C_0^2[0,T]} F(y + \lambda^{-1/2} \mathcal{Z}_{h_1}(x_1, \cdot) + \lambda^{-1/2} \mathcal{Z}_{h_2}(x_2, \cdot)) dm^2(x_1, x_2) \\ &= \int_{C_0^2[0,T]} F(y + \mathcal{Z}_{h_1/\sqrt{\lambda}}(x_1, \cdot) + \mathcal{Z}_{h_2/\sqrt{\lambda}}(x_2, \cdot)) dm^2(x_1, x_2) \\ &= \int_{C_0[0,T]} F(y + \mathcal{Z}_{s(h_1/\sqrt{\lambda}, h_2/\sqrt{\lambda})}(x, \cdot)) dm(x) \\ &= \int_{C_0[0,T]} F(y + \lambda^{-1/2} \mathcal{Z}_{s(h_1,h_2)}(x, \cdot)) dm(x) \\ &= T_{\lambda,s(h_1,h_2)}(F)(y). \end{aligned}$$

Thus we obtain the desired result.  $\square$

Using a mathematical induction and equation (3.8), we obtain the following corollary.

**Corollary 4.6.** Let  $F \in \mathfrak{A}^{(2)}$  be given by equation (4.2), and let  $\mathcal{H} = (h_1, \dots, h_n)$  be a sequence in  $\mathcal{O}_\infty(\mathcal{A})$ . Then for all nonzero real  $q \in \mathbb{R} \setminus \{0\}$ ,

$$(4.7) \quad T_{q,h_n}^{(2)}(\dots(T_{q,h_1}^{(2)}(F))\dots) = T_{q,s(\mathcal{H})}^{(2)}(F)(y)$$

for s-a.e.  $y \in C_0[0, T]$ .

Applying equation (3.7), we also obtain the following corollary.

**Corollary 4.7.** Let  $F \in \mathfrak{A}^{(2)}$  be given by equation (4.2), and let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be sequences in  $\mathcal{O}_\infty(\mathcal{A})$ . Then for all nonzero real  $q \in \mathbb{R} \setminus \{0\}$ ,

$$(4.8) \quad T_{q,s(\mathcal{H}_2)}^{(2)}(T_{q,s(\mathcal{H}_1)}^{(2)}(F))(y) = T_{q,s(\mathcal{H}_1 \wedge \mathcal{H}_2)}^{(2)}(F)(y)$$

for s-a.e.  $y \in C_0[0, T]$ .

## 5. ALGEBRAIC STRUCTURES OF GAUSSIAN-FOURIER-FEYNMAN TRANSFORMS

Given  $q \in \mathbb{R} \setminus \{0\}$ , let

$$\mathbb{T}_{q, \mathcal{O}_\infty(\mathcal{A})} \equiv \mathbb{T}_{q, \mathcal{O}_\infty(\mathcal{A})}[\mathfrak{A}^{(2)}] := \{T_{q,h}^{(2)} : h \in \mathcal{O}_\infty(\mathcal{A}) \cup \{0\}\}$$

denote the class of  $L_2$  analytic Gaussian-FFTs acting on  $\mathfrak{A}^{(2)}$ . In the case that  $h \equiv 0$ , i.e.,  $\|h\|_2 = 0$ , it follows that  $T_{q,0}^{(2)}$  is the identity transform for all  $q \in \mathbb{R}$ . For notational convenience, let  $\mathbf{s}(h) \equiv h$  for  $h \in L_\infty[0, T]$ .

By Theorems 4.4 and 4.5, we see that for all  $h_1, h_2 \in \mathcal{O}_\infty(\mathcal{A}) \cup \{0\}$  and all  $F \in \mathfrak{A}^{(2)}$ ,

$$(T_{q,h_2}^{(2)} \circ T_{q,h_1}^{(2)})(F) \equiv (T_{q,h_2}^{(2)}(T_{q,h_1}^{(2)}(F))) = T_{q,\mathbf{s}(h_1,h_2)}^{(2)}(F)$$

is in  $\mathfrak{A}^{(2)}$ . Because

$$\mathbf{s}(\mathbf{s}(h_3, h_2), h_1) = \mathbf{s}(h_3, h_2, h_1) = \mathbf{s}(h_3, \mathbf{s}(h_2, h_1)),$$

for all  $h_1, h_2, h_3 \in \mathcal{O}_\infty(\mathcal{A}) \cup \{0\}$ , we see that the composition  $\circ$  of  $L_2$  analytic Gaussian-FFTs is associative. Also, because  $\mathbf{s}(h_1, h_2) = \mathbf{s}(h_2, h_1)$ , we see that  $(T_{q,h_2}^{(2)} \circ T_{q,h_1}^{(2)})(F) = (T_{q,h_1}^{(2)} \circ T_{q,h_2}^{(2)})(F)$ , and clearly  $(T_{q,0}^{(2)} \circ T_{q,h}^{(2)})(F) = T_{q,h}^{(2)}(F)$  for any  $h_1, h_2, h \in \mathcal{O}_\infty(\mathcal{A}) \cup \{0\}$  and every  $F \in \mathfrak{A}^{(2)}$ . Thus, we have the following assertion.

**Theorem 5.1.** *The space  $(\mathbb{T}_{q, \mathcal{O}_\infty(\mathcal{A})}, \circ)$  is a commutative monoid. Furthermore, the monoid  $\mathbb{T}_{q, \mathcal{O}_\infty(\mathcal{A})}$  acts on the space  $\mathfrak{A}^{(2)}$  in the sense that  $(T_{q,h}^{(2)}, F) \mapsto T_{q,h}^{(2)}(F)$ .*

Next let  $\mathbf{S}_f(\mathcal{O}_\infty(\mathcal{A}))$  be the set of all finite sequences of functions in  $\mathcal{O}_\infty(\mathcal{A}) \cup \{0\}$ , and let

$$\mathbb{T}_{q, \mathbf{S}_f(\mathcal{O}_\infty(\mathcal{A}))} \equiv \mathbb{T}_{q, \mathbf{S}_f(\mathcal{O}_\infty(\mathcal{A}))}[\mathfrak{A}^{(2)}] := \{T_{q,\mathbf{s}(\mathcal{H})}^{(2)} : \mathcal{H} \in \mathbf{S}_f(\mathcal{O}_\infty(\mathcal{A}))\}.$$

From the fact that for any  $\mathcal{H} \in \mathbf{S}_f(\mathcal{O}_\infty(\mathcal{A}))$ ,  $\mathbf{s}(\mathcal{H}) \in \mathcal{O}_\infty(\mathcal{A}) \cup \{0\} \subset \mathbf{S}_f(\mathcal{O}_\infty(\mathcal{A}))$ , we know that the classes  $\mathbb{T}_{q, \mathcal{O}_\infty(\mathcal{A})}$  and  $\mathbb{T}_{q, \mathbf{S}_f(\mathcal{O}_\infty(\mathcal{A}))}$  coincide as sets. However, we will consider another operation,  $\bar{\wedge}$ , on  $\mathbb{T}_{q, \mathbf{S}_f(\mathcal{O}_\infty(\mathcal{A}))}$ , defined as follows: for  $T_{q,\mathbf{s}(\mathcal{H}_1)}^{(2)}$  and  $T_{q,\mathbf{s}(\mathcal{H}_2)}^{(2)}$  in  $\mathbb{T}_{q, \mathbf{S}_f(\mathcal{O}_\infty(\mathcal{A}))}$ , let

$$T_{q,\mathbf{s}(\mathcal{H}_1)}^{(2)} \bar{\wedge} T_{q,\mathbf{s}(\mathcal{H}_2)}^{(2)} := T_{q,\mathbf{s}(\mathcal{H}_1 \wedge \mathcal{H}_2)}^{(2)},$$

where  $\mathcal{H}_1 \wedge \mathcal{H}_2$  is defined by (3.4) above. By equation (3.5), one can see that the operation  $\bar{\wedge}$  is well defined.

*Remark 5.2.* By equation (4.8), we observe

$$T_{q,\mathbf{s}(\mathcal{H}_1)}^{(2)} \bar{\wedge} T_{q,\mathbf{s}(\mathcal{H}_2)}^{(2)} = T_{q,\mathbf{s}(\mathcal{H}_1 \wedge \mathcal{H}_2)}^{(2)} = T_{q,\mathbf{s}(\mathcal{H}_1)}^{(2)} \circ T_{q,\mathbf{s}(\mathcal{H}_2)}^{(2)}.$$

**Theorem 5.3.** *The space  $(\mathbb{T}_{q, \mathbf{S}_f(\mathcal{O}_\infty(\mathcal{A}))}, \bar{\wedge})$  is a commutative monoid. Furthermore, the monoid  $\mathbb{T}_{q, \mathbf{S}_f(\mathcal{O}_\infty(\mathcal{A}))}$  acts on the space  $\mathfrak{A}^{(2)}$  in the sense that  $(T_{q,\mathbf{s}(\mathcal{H})}^{(2)}, F) \mapsto T_{q,\mathbf{s}(\mathcal{H})}^{(2)}(F)$ .*

*Remark 5.4.* The operation  $\bar{\wedge}$  is a semigroup action of  $\mathbb{T}_{q, \mathbf{S}_f(\mathcal{O}_\infty(\mathcal{A}))}$  on  $\mathfrak{A}^{(2)}$ .

The sequence space  $\mathbf{S}_f(\mathcal{O}_\infty(\mathcal{A}))$  is a monoid under the operation  $\wedge$  given by (3.4). Define an equivalence relation  $\overset{\mathbf{s}}{\sim}$  on  $\mathbf{S}_f(\mathcal{O}_\infty(\mathcal{A}))$  as follows: for  $\mathcal{H}_1$  and  $\mathcal{H}_2$  in  $\mathbf{S}_f(\mathcal{O}_\infty(\mathcal{A}))$ ,

$$\mathcal{H}_1 \overset{\mathbf{s}}{\sim} \mathcal{H}_2 \iff \mathbf{s}(\mathcal{H}_1) = \mathbf{s}(\mathcal{H}_2).$$

Also, let

$$S_f^\sim \equiv S_f(\mathcal{O}_\infty(\mathcal{A})) / \overset{s}{\sim} := \{[\mathcal{H}]_s : \mathcal{H} \in S_f(\mathcal{O}_\infty(\mathcal{A}))\}$$

be the quotient set of  $S_f(\mathcal{O}_\infty(\mathcal{A}))$  by  $\overset{s}{\sim}$ . Then from (3.2) and (3.3), we see that  $S_f^\sim$  is the quotient monoid under the operation  $\wedge$  on  $S_f^\sim$ , given by

$$(5.1) \quad [\mathcal{H}_1]_s \wedge [\mathcal{H}_2]_s := [\mathcal{H}_1 \wedge \mathcal{H}_2]_s.$$

Define a relation on  $\mathbb{T}_{q, S_f(\mathcal{O}_\infty(\mathcal{A}))}$  as follows: for  $T_{q, s(\mathcal{H}_1)}^{(2)}$  and  $T_{q, s(\mathcal{H}_2)}^{(2)}$  in  $\mathbb{T}_{q, S_f(\mathcal{O}_\infty(\mathcal{A}))}$ ,

$$T_{q, s(\mathcal{H}_1)}^{(2)} \overset{t}{\sim} T_{q, s(\mathcal{H}_2)}^{(2)} \iff \mathcal{H}_1 \overset{s}{\sim} \mathcal{H}_2.$$

From (4.7) and (3.3), we see that for every  $(h_1, \dots, h_n) \in S_f(\mathcal{O}_\infty(\mathcal{A}))$  and any permutation  $\pi$  of  $\{1, \dots, n\}$ ,

$$T_{q, s(h_1, \dots, h_n)}^{(2)}(F) = T_{q, s(h_{\pi(1)}, \dots, h_{\pi(n)})}^{(2)}(F)$$

for all  $F$  in  $\mathfrak{A}^{(2)}$ . Thus, the relation  $\overset{t}{\sim}$  is a well-defined equivalence relation, and so we can obtain the quotient monoid

$$\begin{aligned} \mathbb{T}_{q, S_f(\mathcal{O}_\infty(\mathcal{A}))}^\sim &\equiv \mathbb{T}_{q, S_f(\mathcal{O}_\infty(\mathcal{A}))} / \overset{t}{\sim} \\ &:= \{[T_{q, s(\mathcal{H})}^{(2)}]_t : T_{q, s(\mathcal{H})}^{(2)} \in \mathbb{T}_{q, S_f(\mathcal{O}_\infty(\mathcal{A}))}\} \end{aligned}$$

with the operation  $\bar{\wedge}$  given by

$$(5.2) \quad [T_{q, s(\mathcal{H}_1)}^{(2)}]_t \bar{\wedge} [T_{q, s(\mathcal{H}_2)}^{(2)}]_t := [T_{q, s(\mathcal{H}_1 \wedge \mathcal{H}_2)}^{(2)}]_t.$$

**Theorem 5.5.** *The map  $\Xi : (\mathbb{T}_{q, S_f(\mathcal{O}_\infty(\mathcal{A}))}^\sim, \bar{\wedge}) \rightarrow (S_f^\sim, \wedge)$  given by*

$$(5.3) \quad \Xi([T_{q, s(\mathcal{H})}^{(2)}]_t) = [\mathcal{H}]_s.$$

*is a monoid isomorphism.*

*Proof.* It follows from (5.2) and (5.1) that

$$\begin{aligned} \Xi([T_{q, s(\mathcal{H}_1)}^{(2)}]_t \bar{\wedge} [T_{q, s(\mathcal{H}_2)}^{(2)}]_t) &= \Xi([T_{q, s(\mathcal{H}_1 \wedge \mathcal{H}_2)}^{(2)}]_t) \\ &= [\mathcal{H}_1 \wedge \mathcal{H}_2]_s \\ &= [\mathcal{H}_1]_s \wedge [\mathcal{H}_2]_s \\ &= \Xi([T_{q, s(\mathcal{H}_1)}^{(2)}]_t) \wedge \Xi([T_{q, s(\mathcal{H}_2)}^{(2)}]_t). \end{aligned}$$

Clearly, the map given by equation (5.3) is bijective.  $\square$

## 6. FREE GROUP $F(\mathbb{T}_{q, S_f(\mathcal{O}_\infty(\mathcal{A}))}^\sim)$

In this section, we describe a transformation group that is the free group generated by  $\mathbb{T}_{q, S_f(\mathcal{O}_\infty(\mathcal{A}))}^\sim$ .

Given an orthogonal set  $\mathcal{A}$  of functions in  $L_2[0, T]$ , let  $\mathcal{O}_\infty^n(\mathcal{A})$  be the class of all nonzero elements  $h \in L_\infty[0, T]$ , such that  $\mathcal{A}h$  is orthonormal in  $L_2[0, T]$ . For a detailed example of  $\mathcal{O}_\infty^n(\mathcal{A})$ , see Example 4.3 above.

The following lemma is due to Cameron and Storvick in [2, Lemma H].

**Lemma 6.1.** For  $f \in L_2(\mathbb{R}^n)$  and  $h \in \mathcal{O}_\infty^n(\mathcal{A})$ , let  $\psi_{f, \mathcal{A}h}^q(\vec{r})$  be given by equation (4.5). Then  $\psi_{f, \mathcal{A}h}^q \in L_2(\mathbb{R}^n)$ . The integral in the right side of (4.5) is to be interpreted as an  $L_2$ -limiting integral in the sense that

$$\lim_{\delta \rightarrow \infty} \int_{\mathbb{R}^n} \left| \left( \frac{-iq}{2\pi} \right)^{n/2} \int_{-\delta}^{\delta} \dots \int_{-\delta}^{\delta} f(\vec{u}) \exp \left\{ \frac{iq}{2} \sum_{j=1}^n (u_j - r_j)^2 \right\} d\vec{u} - \psi_{f, \mathcal{A}h}^q(\vec{r}) \right|^2 d\vec{r} = 0.$$

In this case, we have  $\|\psi_{f, \mathcal{A}h}^q\|_2 = \|f\|_2$ .

In view of Theorem 4.4 and Lemma 6.1, we obtain the following theorem.

**Theorem 6.2.** Let  $q \in \mathbb{R} \setminus \{0\}$  and let  $h \in \mathcal{O}_\infty^n(\mathcal{A})$ . Then  $L_2$  analytic  $\mathcal{Z}_h$ -FFT,

$$T_{q,h}^{(2)} : \mathfrak{A}^{(2)} \rightarrow \mathfrak{A}^{(2)}$$

is a linear operator isomorphism. That is,  $\|F\|_{\mathfrak{A}^{(2)}} = \|T_{q,h}^{(2)}(F)\|_{\mathfrak{A}^{(2)}}$  for all  $F \in \mathfrak{A}^{(2)}$ .

Thus we have  $\|T_{q,h}^{(2)}\|_o = 1$ , where  $\|\cdot\|_o$  denotes the operator norm.

For any nonzero real  $q$ , let  $\mathbb{T}_{q, \mathcal{S}_f(\mathcal{O}_\infty^n(\mathcal{A}))}^* := \mathbb{T}_{q, \mathcal{S}_f(\mathcal{O}_\infty^n(\mathcal{A}))} \setminus \{[T_{q,0}^{(2)}]_{\mathbf{s}}\}$ . Given  $q \in \mathbb{R} \setminus \{0\}$ , define a map

$$\mathcal{W} : \mathbb{T}_{q, \mathcal{S}_f(\mathcal{O}_\infty^n(\mathcal{A}))}^* \longrightarrow \mathbb{T}_{-q, \mathcal{S}_f(\mathcal{O}_\infty^n(\mathcal{A}))}^*$$

by  $\mathcal{W}([T_{q, \mathbf{s}(\mathcal{H})}^{(2)}]_{\mathbf{t}}) = [T_{-q, \mathbf{s}(\mathcal{H})}^{(2)}]_{\mathbf{t}}$ . Then,  $\mathcal{W}$  is one-to-one correspondence. Thus, by the usual argument in the free group theory, one can obtain the group  $\mathbb{F}(\mathbb{T}_{q, \mathcal{S}_f(\mathcal{O}_\infty^n(\mathcal{A}))}^*)$  freely generated by  $\mathbb{T}_{q, \mathcal{S}_f(\mathcal{O}_\infty^n(\mathcal{A}))}^*$ .

Note that

$$[T_{q, \mathbf{s}(\mathcal{H}_1)}^{(2)}]_{\mathbf{t}} \bar{\wedge} [T_{q, \mathbf{s}(\mathcal{H}_2)}^{(2)}]_{\mathbf{t}} = [T_{q, \mathbf{s}(\mathcal{H}_1 \wedge \mathcal{H}_2)}^{(2)}]_{\mathbf{t}} = [T_{q, \mathbf{s}(\mathcal{H}_1)}^{(2)} \circ T_{\mathbf{s}(\mathcal{H}_2)}^{(2)}]_{\mathbf{t}}.$$

by equation (4.8). Given two transforms  $\mathcal{T}_1$  and  $\mathcal{T}_2$  in  $\mathbb{F}(\mathbb{T}_{q, \mathcal{S}_f(\mathcal{O}_\infty^n(\mathcal{A}))}^*)$ , let the group operation between  $\mathcal{T}_1$  and  $\mathcal{T}_2$  be given by

$$(\mathcal{T}_1 \circ \mathcal{T}_2)(F) \equiv \mathcal{T}_1(\mathcal{T}_2(F)), \quad F \in \mathfrak{A}^{(2)}.$$

For an element  $\mathcal{T}$  of  $\mathbb{F}(\mathbb{T}_{q, \mathcal{S}_f(\mathcal{O}_\infty^n(\mathcal{A}))}^*)$ , let  $l_w(\mathcal{T})$  denote the length of the word  $\mathcal{T}$ . Given  $\mathcal{T} \in \mathbb{F}(\mathbb{T}_{q, \mathcal{S}_f(\mathcal{O}_\infty^n(\mathcal{A}))}^*)$ , assume that  $\mathcal{T}$  is not the empty word (i.e., it is not the identity transform  $[T_{q,0}^{(2)}]_{\mathbf{t}}$ ). If  $l_w(\mathcal{T}) = 1$ , then  $\mathcal{T}$  is an element of the set

$$\mathbb{T}_{q, \mathcal{S}_f(\mathcal{O}_\infty^n(\mathcal{A}))}^* \dot{\cup} \mathbb{T}_{-q, \mathcal{S}_f(\mathcal{O}_\infty^n(\mathcal{A}))}^*.$$

Alternatively, if  $l_w(\mathcal{T}) > 1$ , then  $\mathcal{T}$  cannot be expressed as (an equivalence class of) a single Gaussian-FFT by the concept of the reduced word in the free group theory. But, in view of the assertion (ii) of Theorem 4.4 and Lemma 6.1, we see that for any  $\mathcal{T} \in \mathbb{F}(\mathbb{T}_{q, \mathcal{S}_f(\mathcal{O}_\infty^n(\mathcal{A}))}^*)$ ,  $\mathcal{T}$  is a linear operator isomorphism from  $\mathfrak{A}^{(2)}$  into  $\mathfrak{A}^{(2)}$ .

On the other hand, we consider other algebraic structure of transforms as follows: given  $h \in L_\infty[0, T]$ , let  $T_{0,h}^{(2)}$  denote the identity transform, i.e.,  $T_{0,h}^{(2)}(F) = F$ , and let

$$\mathbb{T}_{\mathbb{R}, h} = \{T_{q,h}^{(2)} : q \in \mathbb{R}\}.$$

Now, by using [7, equation (2.14)] (clearly, equations (2.10) and (2.14) in [7] hold for  $L_2$  analytic Gaussian-FFTs  $T_{q,h}^{(2)}$ ), we obtain the facts that for all  $q_1, q_2 \in \mathbb{R}$  with  $q_1 + q_2 \neq 0$  and all  $F \in \mathfrak{A}^{(2)}$ ,

$$(T_{q_2,h}^{(2)} \circ T_{q_1,h}^{(2)})(F) \equiv T_{q_2,h}^{(2)}(T_{q_1,h}^{(2)}(F)) \approx T_{\frac{q_1 q_2}{q_1 + q_2}, h}^{(2)}(F),$$

and

$$(T_{q_2,h}^{(2)} \circ T_{q_1,h}^{(2)})(F) \approx (T_{q_1,h}^{(2)} \circ T_{q_2,h}^{(2)})(F).$$

If  $q_1 + q_2 = 0$ , then by Theorem 4.4, we obtain  $T_{q_2,h}^{(2)} = T_{-q_1,h}^{(2)} = \{T_{q_1,h}^{(2)}\}^{-1}$ .

**Theorem 6.3.** *For each  $h \in L_\infty[0, T]$ , the space  $(\mathbb{T}_{\mathbb{R},h}, \circ)$  forms a commutative group. Clearly,  $\mathbb{T}_{\mathbb{R},h}$  with  $\|h\|_2 = 0$  is a trivial group.*

We note that given  $h \in L_\infty[0, T]$  with  $\|h\|_2 > 0$ , the transformation group  $\mathbb{T}_{\mathbb{R},h}$  is the free group with the free basis  $\{T_{q,h}^{(2)} : q > 0\}$ . Because

$$|\{T_{q,h}^{(2)} : q > 0\}| = \aleph_1 = \aleph_0^{\aleph_0} = |\mathbb{S}_f| \equiv |\mathbb{T}_{q,\mathbb{S}_f(\mathcal{O}_\infty^n(\mathcal{A}))}^{\sim}|,$$

the free groups  $F(\mathbb{T}_{q,\mathbb{S}_f(\mathcal{O}_\infty^n(\mathcal{A}))}^{\sim})$  and  $\mathbb{T}_{\mathbb{R},h}$  have the same rank. Thus, by the concept of rank of free groups [16, Proposition 2.1.4, p.47], we conclude that

$$F(\mathbb{T}_{q,\mathbb{S}_f(\mathcal{O}_\infty^n(\mathcal{A}))}^{\sim}) \stackrel{\iota}{\cong} \mathbb{T}_{\mathbb{R},h},$$

where  $\iota$  is a group isomorphism.

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